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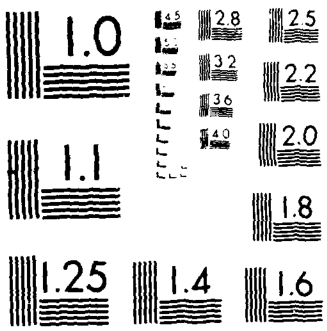
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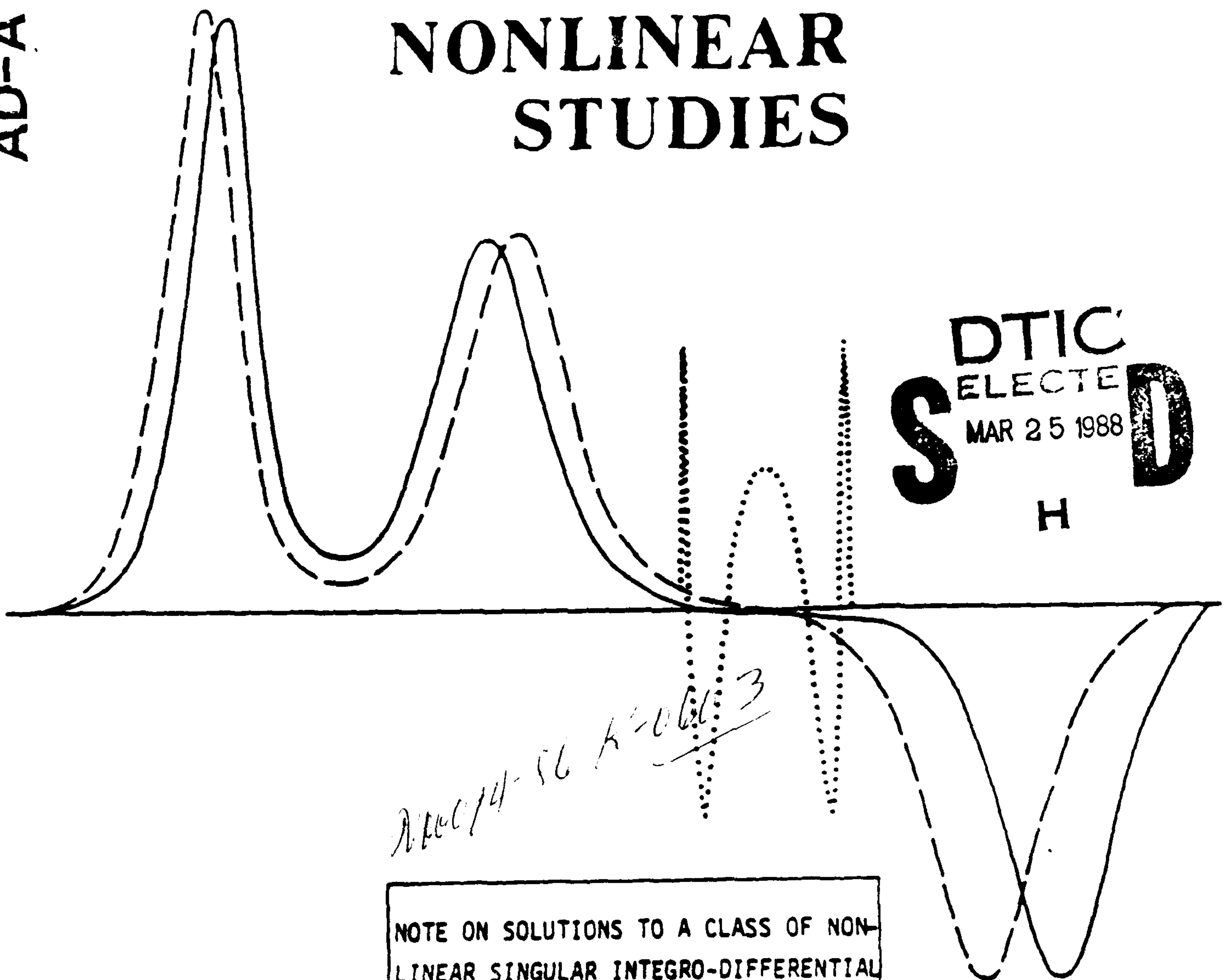

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NOTE ON SOLUTIONS TO A CLASS OF NONLINEAR SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS

by

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In recent years considerable interest has focused on certain physically important nonlinear evolution equations which can be linearized. Many of these equations fall into the category of linearization via soliton theory and the Inverse Scattering Transform (IST) (for a review of much of this work, see for example [1]). Well-known equations are the Korteweg-deVries Equation (KdV)

$$u_t + 2uu_x + u_{xxx} = 0, \tag{1}$$

the sine-Gordon equation

$$u_{xt} = \sin u , \tag{2}$$

and the Kadomtsev-Petviashvili (KP) equation

$$(u_t + 2uu_x + u_{xxx})_x = -3\sigma^2 u_{yy} . \tag{3}$$

Each of these equations has certain singular integro-differential analogs, the best known being the so-called Benjamin-Ono equation

$$u_t + 2uu_x + (Hu)_{xx} = 0 . \tag{4}$$

Analoques of the sine-Gordon and of the KP equations include the sine-Hilbert equation,

$$(Hu)_t = \sin u \tag{5}$$

and

$$(u_t + Hu_{xx} + 2uu_x)_x = (Hu_{xx} + 2uu_x)_y , \tag{6}$$

respectively. In the above, Hu is the Hilbert transform of u.

This work was motivated by some recent results of Constantin, Lax, and Majda [6]; in particular these authors proposed the following equation as a model for the motion of vorticity for an inviscid incompressible fluid flow,

$$u_t = uHu. \tag{12}$$

They introduced the transformation (10) and showed that w satisfies the ODE,

$$w_t = -\frac{i}{2}w^2. \tag{13}$$

We first consider A). It should be noted that the above result can be obtained as follows. Operate on (12) with $(1 + iH)$ and use

$$H(uHu) = ((Hu)^2 - u^2)/2, \tag{14}$$

which is a special case of the known formula

$$H(fHg) + H(gHf) = (Hf)(Hg) - fg. \tag{15}$$

The above result can easily be extended. Since as is known $H^2 = -1$ we have that $Hw = -iw$. Now w is the boundary value of a function analytic in the lower half plane (a "lower function"), vanishing at infinity. Hence, $Hw = -iw$, and more generally,

$$w = u+iHu \implies Hw^n, \quad (n > 0,\ \text{integer}). \tag{16}$$

$$He^w = -ie^w \tag{17}$$

etc. This enables us to construct arbitrarily many reducible equations such as, Let us consider the initial value problem for each of the above equations with u real. Given $u(x,0)$, initial values for $w(x,t)$ are obtained from $w(x,0) = u(x,0) + iHu(x,0)$, and the solution $u(x,t)$ is recovered from $u(x,t) = \mathrm{Re}\, w(x,t)$.

Generalizations to systems of equations as well as discrete analogs are immediate using these ideas, hence we shall not incorporate them into this discussion. Multidimensional analogs can also be readily constructed. For example, an analog to the K-P equation (3) is

$$\frac{\partial}{\partial x}(u_t + u_{xxx} + (2uHu)_x) = -3\sigma^2 u_{yy} , \tag{19}$$

σ^2 = const and is linearized via the KP euqation

$$\frac{\partial}{\partial x}(w_t + w_{xxx} - i(w^2)_x) = -3\sigma^2 w_{yy}, \tag{20}$$

which is formally a rescaled version of (3). Equation (19) is (2+1)-dimensional. A (3+1)-dimensional equation can also be linearized via (20). Namely let $H_z u$ denote the Hilbert transform of $u(x,y,z,t)$ with respect to the variable z, i.e.,

$$H_z u = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(x,y,\xi,t)}{\xi - z} d\xi \quad . \tag{21}$$

Then instead of K-P: we may consider a multi-dimensional analog of (19):

$$\frac{\partial}{\partial x}(u_t + u_{xxx} + 2(uH_z u)_x) = -3\sigma^2 u_{yy}, \tag{22}$$

and it is also mapped to the KP equation (20), via $w = u + iH_z u$.

$$U_t = UV,\ U_x = V_y,\ U_y = -V_x,\quad y \le 0. \qquad (28)$$

Equations (12), (10) are special cases of the above, $u(x,t) = U(x,0,t)$, $v(x,t) = V(x,0,t)$. We note that these equations are mutually consistent. However, it should be stressed that equations (28) are not a (2+1)-dimensional system since the latter two equations in (28) are the Cauchy-Riemann equations and so $W = U + iV = W(z,t)$, $z = x + iy$. The transformation $W = U + iV$ maps (28) to

$$W_t = -\frac{i}{2} W^2 + C(t) \qquad (29)$$

This is derived as follows:

$$V_{yt} = U_{xt} = \frac{\partial}{\partial x}(UV),$$

$$V_{xt} = -U_{yt} = -\frac{\partial}{\partial y}(UV). \qquad (30)$$

Using the formula $g(x,y) = \int g_x dx + g_y dy$, from equations (28) and (30) we obtain

$$V_t = \int \{-\frac{\partial}{\partial y}(UV)dx + \frac{\partial}{\partial x}(UV)dy\}$$

$$= \int (VV_x - UU_x)dx + (V_yV - UU_y)dy$$

$$= \frac{1}{2}(V^2 - U^2) - iC(t)\ .$$

Hence

$$W_t = -\frac{i}{2} W^2 + C(t).$$

From the above discussion it follows that the results of (A)-(C) are also valid if one replaces H by T or by another suitable operator (see [13]).

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